Order Recursive Gaussian Elimination and Efficient CAD of Microwave Circuits

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Abstract. An order-recursive variant of Gaussian elimination has been presented for efficient solution of linear equations resulting from augmenting the feed line impedence matrix by block row and column vectors corresponding to reactions associated with discontinuities in the moment method simulation of MMIC elements. The potential utility of the solution technique in a CAD environment is demonstrated by applying it to the interactive design of microstrip low-pass filter.

1. Introduction

The well-known method of moments (MoM) is widely used in the simulation and computer-aided design (CAD) of microwave and millimeter-wave integrated circuits (MMICs) (cf. [1], [2]). In MoM, the boundary value problem for the unknown current distribution over the surface of the conductors is formulated as an electrical field integral equation (EFIE). The EFIE is then converted into a system of linear algebraic equations (for the current) by the application of suitable basis and testing functions. The circuit characteristics, such as S-parameters, radiation and metallization losses, etc., can be derived from the computed current distribution.

The system (or moment) matrix that represents the relationship between the basis and the test elements used to solve for the current distribution, is typically dense. For moderately high-order models ($\mathcal{O}(200-300)$), the current distribution may be obtained as the solution of a system of linear algebraic equations using LU decomposition and subsequent solution of two triangular systems of equations. The computational complexity of the solution of system of equations of order N is N^3. For several applications in circuit modeling, where N is fixed, use of conventional method (via LU decomposition) of solution of system of equations is adequate and provides the most efficient means for computation.

However, in certain situations, the order of systems of equations to be solved may change from N to $N+M$, where the original $(N \times N)$ data matrix becomes a submatrix of the higher-order $(N+M) \times (N+M)$ matrix as a result of augmenting the model. This is frequently encountered in characterization of MMIC elements where the data matrix is recursively augmented with new row and column vectors that correspond to circuit extensions, stubs, etc. At present, each augmented matrix is treated as a new data matrix and the solution of augmented system of equations is recomputed from scratch. The resulting solution procedure becomes computationally inefficient, and, as shown later, in the worst case, the computational complexity can become $\mathcal{O}((N+M)^4)$. The primary objective of this paper is to apply a variant of Gaussian elimination, called *order recursive Gaussian elimination* (ORGE) [3], to develop a solution procedure of computational complexity $\mathcal{O}((N+M)^3)$. This order of magnitude reduction in computations is clearly very attractive for the CAD of MMICs.

2. Augmented Matrix Model of MMIC

2.1. Discontinuity Analysis

As an example of MMIC analysis where augmented matrices discussed earlier occur, consider the two-port microstrip discontinuity shown in Fig. 1(a).

The input and output ports are connected by transmission lines at reference planes 1 and 2 to shunt coaxial terminations (at $1'$ and $2'$). Application of MoM to this circuit yields total currents on the whole structure, including the connection lines. It is required to calculate complex amplitude of the incident and reflected currents at planes 1 and 2 by discarding the influence of port connection lines and coax excitation – a process known as de-embedding. Note that the discontinuity in Fig. 1(a) is excited by coaxial cables connected to de-embedding transmission lines of length L. De-embedding requires the solution of additional subproblems such as the cascade of the input and output lines (Fig. 1(b)) and a line terminated in a short circuit (Fig. 1(c)) to transform the open-circuit impedance (or Z-parameters) matrix from planes $(1', 2')$ to $(1, 2)$ [4]. Alternatively, one could determine the complex amplitude of the incident wave at port 1 from the transmission line model in Fig. 1(b) and embed it in the solution of Fig. 1(a) to compute the S-parameters [5]. The latter approach is used to arrive at the augmented matrix model of the circuit.

Assume that the input and output lines (of length L) support N_i and N_o basis elements, and the discontinuity (between planes 1 and 2) supports N_d basis elements. Then,

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the matrix equations resulting from application of MoM to the circuit elements in Fig. 1(a) and 1(b), respectively, can be written as:

$$\begin{bmatrix} \mathbf{Z}^{ii} & \mathbf{Z}^{id} & \mathbf{Z}^{io} \\ \mathbf{Z}^{di} & \mathbf{Z}^{dd} & \mathbf{Z}^{do} \\ \mathbf{Z}^{oi} & \mathbf{Z}^{od} & \mathbf{Z}^{oo} \end{bmatrix} \begin{bmatrix} \mathbf{I}^{i} \\ \mathbf{I}^{d} \\ \mathbf{I}^{o} \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{i}_{inc} \\ \mathbf{V}^{d}_{inc} \\ \mathbf{V}^{o}_{inc} \end{bmatrix} \quad (1)$$

$$\begin{bmatrix} \mathbf{Z}^{ii} & \mathbf{Z}^{io} \\ \mathbf{Z}^{oi} & \mathbf{Z}^{oo} \end{bmatrix} \begin{bmatrix} \mathbf{I}^{iL} \\ \mathbf{I}^{oL} \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{i}_{inc} \\ \mathbf{V}^{o}_{inc} \end{bmatrix} \quad (2)$$

Knowing the port currents from the solution of (1) and (2), the two independent S-parameters for a symmetrical two-port may be computed as

$$S_{11} = -\frac{I_1^i - I_1^{iL}}{I_1^{iL}}, \qquad S_{21} = \frac{I_2^o}{I_1^{iL}}. \quad (3)$$

It is evident from (1) and (2) that the system matrix in (2) is a submatrix of the matrix in (1). Therefore, if (2) is solved first, it should be possible to embed its solution in (1) to make the solution of (1) more efficient. Note that the block submatrices of the system matrix in (1) may be rearranged to get

$$\left[\begin{array}{cc|c} \mathbf{Z}^{ii} & \mathbf{Z}^{io} & \mathbf{Z}^{id} \\ \mathbf{Z}^{oi} & \mathbf{Z}^{oo} & \mathbf{Z}^{od} \\ \hline \mathbf{Z}^{di} & \mathbf{Z}^{do} & \mathbf{Z}^{dd} \end{array}\right] \begin{bmatrix} \mathbf{I}^{i} \\ \mathbf{I}^{o} \\ \mathbf{I}^{d} \end{bmatrix} = \begin{bmatrix} \mathbf{V}^{i}_{inc} \\ \mathbf{V}^{o}_{inc} \\ \mathbf{V}^{d}_{inc} \end{bmatrix} \quad (4)$$

Clearly, the system (4) is a bordered matrix, obtained by augmenting the system matrix of (2), and is referred to as *augmented* matrix model of the discontinuity. In the sequel, it will be discussed how the augmented matrix equation can be solved efficiently.

2.2. Interactive Filter Design

The use of ORGE will provide further efficiency in MMIC design applications. Consider the double-stub low-pass filter shown in Fig. 2. The input and output lines indicated by ports 1 and 2 are oriented longitudinally. It is desired to design two stubs (each of length Y_S) such that the filter has a prespecified cut-off frequency and pass-band roll-off. It should be pointed out that, although the desired filter can be obtained using circuit simulation, we have considered this simple example to illustrate the computational advantages of ORGE in microwave CAD.

Assume that the input and output transmission lines are gridded such that they support N_i and N_o basis elements, respectively. On applying the MoM, the cascade of these two lines generates a moment matrix which we will refer to as *line matrix.* The line matrix and the corresponding system of equations for computing line currents are identical to eq. (2). Next, we add two stubs symmetrically on either side of the line (see Fig. 2) and iteratively increase their length until the filter response (insertion loss) displays the specified cut-off frequency and the pass-band roll-off. During each iteration, an increasingly large-order linear system of equations needs to be solved to verify whether or not the desired response has been achieved. Assume that the stub length is increased successively in M steps, where each iteration contributes additional N_d rows and columns to the system matrix. Then, at the r-th iteration, the system matrix will be of $(N_i + N_o + rN_d)$-th order, $r = 1, \ldots, M$. When the inter-relation of stubs with lines is accounted for, the resulting system matrix at M-th iteration will have the following structure:

$$\left[\begin{array}{cc|ccccc} \mathbf{Z}^{ii} & \mathbf{Z}^{io} & \mathbf{Z}^{id_1} & \cdots & \mathbf{Z}^{id_r} & \cdots & \mathbf{Z}^{id_M} \\ \mathbf{Z}^{oi} & \mathbf{Z}^{oo} & \mathbf{Z}^{od_1} & \cdots & \mathbf{Z}^{od_r} & \cdots & \mathbf{Z}^{od_M} \\ \hline \mathbf{Z}^{d_1 i} & \mathbf{Z}^{d_1 o} & \mathbf{Z}^{d_1 d_1} & \cdots & \mathbf{Z}^{d_1 d_r} & \cdots & \mathbf{Z}^{d_1 d_M} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{Z}^{d_r i} & \mathbf{Z}^{d_r o} & \mathbf{Z}^{d_r d_1} & \cdots & \mathbf{Z}^{d_r d_r} & \cdots & \mathbf{Z}^{d_r d_M} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \mathbf{Z}^{d_M i} & \mathbf{Z}^{d_M o} & \mathbf{Z}^{d_M d_1} & \cdots & \mathbf{Z}^{d_M d_r} & \cdots & \mathbf{Z}^{d_M d_M} \end{array}\right] \quad (5)$$

with appropriate voltage vector and the unknown current distribution vector. The system in (5) represents the augmented matrix model of the design.

3. Order Recursive Gaussian Elimination

In this section, an order recursive variant of conventional Gaussian elimination method for computation of $\mathbf{x}_r$ in

$$\mathbf{A}_r \mathbf{x}_r = \mathbf{b}_r$$

is presented. The proposed algorithm is developed for the case when *all* the leading principal submatrices of the infinite dimensional matrix $\mathbf{A}$ are nonsingular. Therefore, the solution (albeit suboptimal) can always be computed without the need of pivoting. Assume that the Gaussian elimination on $(\mathbf{A}_r, \mathbf{b}_r)$ has been computed and that for $(\mathbf{A}_{r+1}, \mathbf{b}_{r+1})$ is desired.

3.1. The ORGE Algorithm

The operations of order recursive Gaussian elimination can be divided into two major categories.

1. The first category consists of updating the new column of the augmented matrix $\mathbf{A}_{r+1}$ such that the effect of row operations performed on the matrix $\mathbf{A}_r$ is reflected in the $(r+1)$-th column of $\mathbf{A}_{r+1}$ and
2. The second category consists of the elimination of the elements of $(r+1)$-th row to transform the augmented pair $[\mathbf{A}_{r+1}, \mathbf{b}_{r+1}]$ to an upper trapezoidal form.

General steps for performing these operations are discussed next.

Updating $\mathbf{A}_{r+1}$

At the $(r+1)$-th step, assume that following the recursive Gaussian elimination procedure mentioned above the pair

$[\mathbf{A}_r \mid \mathbf{b}_r]$ has been reduced to an upper trapezoidal form. On augmenting $\mathbf{A}_r$ by an additional row and column and $\mathbf{b}_r$ by an additional row, we have
$[\mathbf{A}_{r+1}|\mathbf{b}_{r+1}] =$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1,r+1} & b_1 \\ \eta_{21} & \hat{a}_{22} & \cdots & a_{2,r+1} & \hat{b}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \eta_{r,1} & \eta_{r,2} & \cdots & a_{r,r+1} & \hat{b}_r \\ a_{r+1,1} & a_{r+1,2} & \cdots & a_{r+1,r+1} & b_{r+1} \end{array}\right] \quad (6)$$

To update the $(r+1)$-th column of $\mathbf{A}_{r+1}$, set

$$\begin{bmatrix} \hat{a}_{2,r+1} \\ \vdots \\ \hat{a}_{r,r+1} \end{bmatrix} := \begin{bmatrix} a_{2,r+1} \\ \vdots \\ a_{r,r+1} \end{bmatrix} - a_{1,r+1} \begin{bmatrix} \eta_{21} \\ \vdots \\ \eta_{r,1} \end{bmatrix}.$$

where $:=$ denotes assignment. This transformation accounts for elimination of the first column of $\mathbf{A}_r$ using a_{11} as the pivot. Next, to account for the operations performed for eliminating $a_{3,2}$ to $a_{3,r}$ using $\hat{a}_{22}$ as the pivot, set

$$\begin{bmatrix} \hat{a}_{3,r+1} \\ \vdots \\ \hat{a}_{r,r+1} \end{bmatrix} := \begin{bmatrix} \hat{a}_{3,r+1} \\ \vdots \\ \hat{a}_{r,r+1} \end{bmatrix} - \hat{a}_{2,r+1} \begin{bmatrix} \eta_{32} \\ \vdots \\ \eta_{r,2} \end{bmatrix}.$$

This is continued until

$$\hat{a}_{r,r+1} := \hat{a}_{r,r+1} - \hat{a}_{r-1,r+1}\eta_{r,r-1}$$

at which point the $(r+1)$-th column of $\mathbf{A}_{r+1}$ is fully updated.

Triangularizing $\mathbf{A}_{r+1}$

Now, using $\hat{a}_{i,i}$, $i = 1, \ldots, r$, as pivots, the elements $a_{r+1,1}$ to $a_{r+1,r}$ are reduced to zero and $a_{r+1,r+1}$ and b_{r+1} are appropriately modified. Specifically, to eliminate the $(r+1, 1)$ element of updated $\mathbf{A}_{r+1}$, set,

$$\begin{aligned} \hat{a}_{r+1,[1\cdots r+1]} &:= \hat{a}_{r+1,[1\cdots r+1]} - \eta_{r+1,1}\hat{a}_{1,[1\cdots r+1]} \\ \hat{b}_{r+1} &:= b_{r+1} - \eta_{r+1,1}b_1 \end{aligned}$$

and in general, to eliminate $(r+1, k)$-th element, set

$$\begin{aligned} \hat{a}_{r+1,[k\cdots r+1]} &:= \hat{a}_{r+1,[k\cdots r+1]} - \eta_{r+1,k}\hat{a}_{k,[k\cdots r+1]} \\ \hat{b}_{r+1} &:= \hat{b}_{r+1} - \eta_{r+1,k}b_k \end{aligned} \quad (7)$$

Note that in (7), due to the assumption of non-singularity of the leading principal submatrices, the element $\hat{a}_{r+1,r+1}$ of the transformed matrix $\mathbf{A}_{r+1}$ is not 0, hence, the recursion can be continued. Based on the discussion in the preceding paragraphs, a formal algorithm for order recursive Gaussian elimination (without pivoting) can be easily formulated [3].

3.2. Computational Complexity

It is clear from the description in previous section that the computational complexity of ORGE is identical to Gaussian elimination. The only difference is the sequence in which the elimination is performed. Hence in ORGE, the elimination and back substitution together require $\mathcal{O}(N+M)^3$ operations, where $(N+M)$ is the dimension of the final augmented matrix. However, if a new Gaussian elimination together with back substitution is performed for each system of order $(N+iN_d)$, $i = 0, \ldots, r$, where N_d is the number of rows and columns by which the matrix is augmented in each iteration and $(rN_d) = M$, then the operations count is approximately $\sum_{i=0}^{r}(N+iN_d)^3$. To see that the latter is an order of magnitude higher than the former, consider the following example.

Let $N = 100$, $N_d = 10$, $r = 10$, i.e., we are augmenting the system matrix by 10 rows and columns in each iteration and performing a total of 10 iterations. The resulting operations count is shown in Fig. 3. Note that since the intent is to observe the order or magnitude complexity, the operations count has been weighted down by $(N+iN_d)^3$ for both methods at each iteration. It can be seen clearly that while ORGE operations count exhibits a slope of 0 (implying $\mathcal{O}(N+M)^3$ operations count), as anticipated, the count for the case when the entire solution is recomputed from scratch shows constant nonzero slope.

4. Simulation Results

Fig. 4 shows the insertion loss at a few iterations for a microstrip double-stub, low-pass filter (shown in Fig. 2) on alumina substrate ($\epsilon_r = 9.9$, $h = 0.127$ mm). The metalization is 5 microns thick copper ($\sigma = 5.8 \times 10^7$ S/m). The stub length is increased in 5 iterations form $Y_S = 1.458$ mm to $Y_S = 2.916$ mm. The current distribution over the filter is computed at each iteration using an efficient PC-based moment method implementation described in [6], which employs interpolated Green's functions and exploits symmetries and redundancies in the various reactions to fill the moment matrix. Once filled, the system of linear equations is solved using the ORGE algorithm. The S-parameters are computed at the reference planes located on the input and output lines at a distance of 2 mm from each filtering stub.

Fig. 3 clearly shows that the designed cut-off frequency of 10 GHz and the passband roll-off of about 40 dB are achieved after the fifth iteration. Computationally, instead of solving the linear systems of order $(N_i + N_o + iN_d)$, $i = 0, 1, \ldots, 5$, which involves $\mathcal{O}(M(N_i + N_o + M)^3)$ operations, using ORGE, we effectively solved one $(N_i + N_o + M)$-th order system with $\mathcal{O}((N_i + N_o + M)^3)$ operations. For the present simulation, $N_i = N_o = 20$, $N_d = 10$ and $r = 5$. The solution of linear system of equations using ORGE required 7.5×10^5 complex operations, compared to 2.6×10^6 required for solving the complete problem from scratch at each iteration.

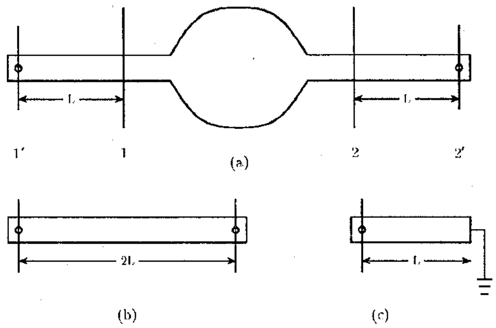

Fig. 1(a). A two-port microstrip discontinuity.
1(b). De-embedding line cascaded to its mirror image.
1(c). De-embedding line terminated in a short circuit.

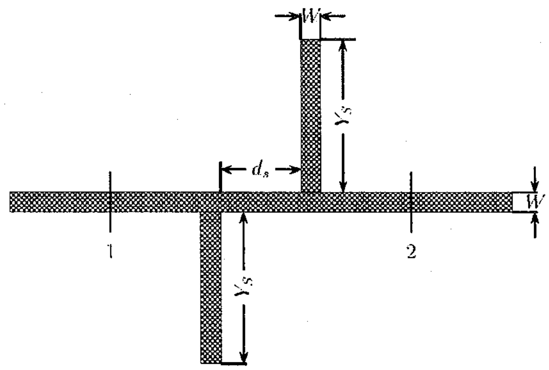

Fig. 2. Microstrip double-stub low-pass filter.

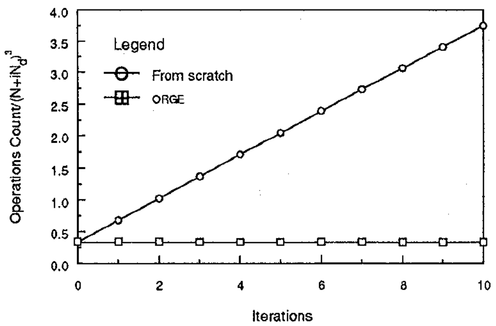

Fig. 3. Order of magnitude estimate of operations count.

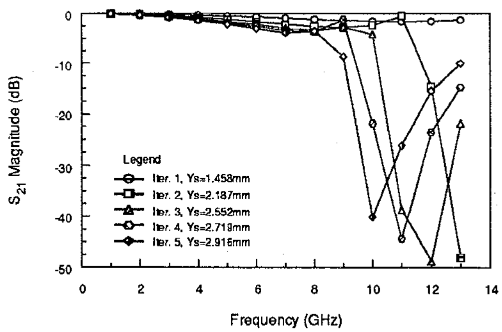

Fig. 4. Iterative double-stub low-pass filter design.

5. Concluding Remarks

In this paper, an order-recursive variant of Gaussian elimination has been presented for efficient solution of linear equations resulting from augmenting the feed line impedance matrix by block row and column vectors corresponding to reactions associated with discontinuities in the moment method simulation of MMIC elements. The usefulness of ORGE in a CAD environment was demonstrated by its application to the design of a microstrip double-stub low-pass filter. ORGE speeds up de-embedding the circuit parameters of discontinuities by up to a factor of N/c – where N is the order of the circuit model and c is a constant considerably smaller than N. It is anticipated to be very useful in the simulation, optimization and computer aided design of microwave circuits.

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